for various values of the parameter $\alpha = \varepsilon/(mgl)$. The solid lines represent stable solutions and the dashed lines unstable ones. It can be seen that as α increases the length of the interval (ω_*, ω_y), corresponding to stable solutions with amplitude $A < \pi$, increases. The amplitude behaviour of the solutions under consideration, which depends on the dimensionless frequency η and the parameter α characterizing the magnitude of the forcing term, completely agrees with the theoretical results established above.

As can be seen from Fig.3, for sufficiently small α $(\alpha < \alpha_* \simeq 3)$ there is a frequency interval in which there exists a second solution $x_2(t, \varepsilon)$ of the type under consideration with amplitude $A_2 < \pi$, coinciding with $x(t, \varepsilon)$ for $\omega = \omega_*$. The solution $x_2(t, \varepsilon)$ is unstable; unlike $x(t, \varepsilon)$ its amplitude increases with ω and decreases with ε.

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THE PERTURBED MOTIONS OF A SOLID CLOSE TO REGULAR LAGRANGIAN PRECESSIONS*

V.V. SAZONOV and V.V. SIDORENKO

The asymptotic behaviour of a Lagrange gyroscope under the influence of a weak perturbing moment is investigated for the case of motions that are close to regular precessions. An averaged system of equations of motion is obtained in special evolutionary variables. The cases of a small constant moment and the presence of a cavity filled with highly viscous fluid are considered in detail.

1. ***The equations of motion and statement of the problem.*** The motion of a heavy axisymmetric rigid body with a fixed point on the axis of symmetry (a Lagrange gyroscope) under the influence of a perturbing mechanical moment of arbitrary nature is described by the equations

$$\Omega_1^{\cdot} = -(\lambda\Omega_3 - \Omega_2 \operatorname{ctg}\vartheta)\Omega_2 + \varkappa \sin\vartheta + \varepsilon M_1 \tag{1.1}$$
$$\Omega_2^{\cdot} = (\lambda\Omega_3 - \Omega_2 \operatorname{ctg}\vartheta)\Omega_1 + \varepsilon M_2$$

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$$\lambda\Omega_3^{\cdot} = \varepsilon M_3,\ \psi^{\cdot} = \Omega_2/\sin\vartheta,\ \vartheta^{\cdot} = \Omega_1,\ \varphi^{\cdot} = \Omega_3 - \Omega_2 \operatorname{ctg}\vartheta$$

Here ψ, ϑ and φ are the standard Euler angles, differentiation with respect to time t is denoted by the dot, λ is the ratio of the body's axial and equatorial moments of inertia. $\varkappa$ is ratio of the product of the mass of the body, the acceleration due to gravity and the distance of the fixed point from the centre of mass, to the equatorial moment of inertia. Ω_i and εM_i $(i = 1, 2, 3)$ are the projections onto the Rezal axes /1/ of the angular velocity vector and the perturbing mechanical moment relative to the equatorial moment of inertia, and ε is a small parameter. If the projections of the above vectors on to the principal axes of inertia of the body are denoted by ω_i and εm_i, then

$$\Omega_1 = \omega_1 \cos\varphi - \omega_2 \sin\varphi,\ \Omega_2 = \omega_1 \sin\varphi + \omega_2 \cos\varphi,\ \Omega_3 = \omega_3$$
$$M_1 = m_1 \cos\varphi - m_2 \sin\varphi,\ M_2 = m_1 \sin\varphi + m_2 \cos\varphi,\ M_3 = m_3$$

Eqs.(1.1) appear in the analysis of many applied problems /1, 2/. The purpose of this paper is to obtain solutions of these equations that are close to regular precession. The method of investigation is the averaging of Eqs.(1.1) over the unperturbed motion and the use of special evolutionary variables. Another method of averaging along the Lagrangian motion was used in /2-4/.

2. *Evolutionary variables.* We consider unperturbed motion. For $\varepsilon = 0$ the general solution of system (1.1) is expressed in terms of the Jacobi elliptic functions /5/, with Ω_3 independent of time, Ω_1, Ω_2, and ϑ depending periodically on t with period T_ϑ, and ψ and φ having the form

$$\psi = \omega_\psi t + \psi_1(t),\ \varphi = \omega_\varphi t + \varphi_1(t) \tag{2.1}$$

where ψ_1 and φ_1 are T_ϑ-periodic functions of t. The frequencies $\omega_\vartheta = 2\pi/T_\vartheta$, ω_ψ and ω_φ depend in a complicated manner on the values of the first integrals of system (1.1) with $\varepsilon = 0$;

$$\begin{gathered} R = \lambda\Omega_2,\ L = \Omega_2 \sin\vartheta + \lambda\Omega_3 \cos\vartheta \\ E = (\Omega_1^2 + \Omega_2^2 + \lambda\Omega_3^2)/2 + \varkappa\cos\vartheta \end{gathered} \tag{2.2}$$

System (1.1) has a four-parameter family of stationary solutions

$$\begin{gathered} \Omega_1 \equiv 0,\ \Omega_2 \equiv \Omega_{20},\ \Omega_3 \equiv W,\ \vartheta \equiv \Theta \\ \psi = \omega_{\psi 0} t + \psi_0,\ \varphi = \omega_{\varphi 0} t + \varphi_0 \end{gathered} \tag{2.3}$$

in which the constants ψ_0 and φ_0 are arbitrary, and the constants $\omega_{\psi 0}, \omega_{\varphi 0}, \Omega_{20}, W$ and Θ are connected by the relations

$$\begin{gathered} \Omega_{20}^2 \operatorname{ctg}\Theta - \lambda W \Omega_{20} + \varkappa \sin\Theta = 0 \\ \omega_{\psi 0} = \Omega_{20}/\sin\Theta,\ \omega_{\varphi 0} = W - \Omega_{20}\operatorname{ctg}\Theta \end{gathered} \tag{2.4}$$

These solutions are called regular precessions.

Choosing as parameters for the family of precessions the quantities W, Θ, ψ_0 and φ_0, from relations (2.4) we obtain

$$\Omega_{20}(W, \Theta) = \operatorname{tg}\Theta\,(\lambda W \pm \sqrt{d})/2,\quad d = \lambda^2 W^2 - 4\varkappa\cos\Theta \tag{2.5}$$

Thus, in the general case, two regular precessions are possible, corresponding to different signs in (2.5). From now on any expression that contains Ω_{20} will be assumed to refer simultaneously to both precessions.

In the unperturbed system the variables ψ and φ are cyclic, and so for $\varepsilon = 0$ a closed subsystem of equations for Ω_1, Ω_2, and ϑ can be extracted from (1.1), containing Ω_3 as a parameter. In the phase space $(\Omega_1, \Omega_2, \vartheta)$ we consider an integral manifold $S_{W,\Theta}$ with a fixed value for the integral L, pertaining to regular precession with parameters W and Θ /6/. Its parametric representation has the form

$$\begin{gathered} S_{W,\Theta} = \{(\Omega_1, \Omega_2, \vartheta) : \Omega_1 = \Omega_1(W, \Theta, c, \nu),\ \Omega_2(W, \Theta, c, \nu), \\ \vartheta = \vartheta(W, \Theta, c, \nu),\ 0 \leqslant \nu \leqslant 2\pi,\ 0 \leqslant c \leqslant c_0(W, \Theta)\} \end{gathered} \tag{2.6}$$

where c and ν stand for the amplitude and phase of the nutational oscillations. At individual solutions lying on the manifold $S_{W,\Theta}$, $\nu = \omega_\vartheta t + \nu_0$. Using Lyapunov's holomorphic integral theorem /7/, (to apply Lyapunov's theorem it is necessary to reduce the order of the system using the integral $L = \text{const}$), the functions Ω_1, Ω_2 and ϑ can be expressed in the form of the series

$$\Omega_1 = \sum_{k=1}^{\infty} c^k \Omega_{1k}(W, \Theta, \nu), \quad \Omega_2 = \Omega_{20}(W, \Theta) + \sum_{k=1}^{\infty} c^k \Omega_{2k}(W, \Theta, \nu) \tag{2.7}$$
$$\vartheta = \Theta + \sum_{k=1}^{\infty} c^k \vartheta_k(W, \Theta, \nu)$$

which converge for sufficiently small values of $|c|$. In particular, $\vartheta = \Theta + c\cos\nu + c^2(\operatorname{ctg}\Theta - 2\varkappa\sin\Theta/\omega_*^2)(3 - 2\cos\nu - \cos 2\nu) + O(c^3)$.

Here $\omega_* = \sqrt{\Omega_{20}^2 + d}$ is the frequency of small nutational oscillations.

In system (1.1) in the neighbourhood of the solutions (2.3) we shall make the local change of variables $(\Omega_1, \Omega_2, \Omega_3, \vartheta, \psi, \varphi) \to (W, \Theta, c, \nu, \psi, \varphi)$ defined by relations (2.7). The new variables have a simple mechanical meaning: W and Θ distinguish the basic regular precession, while c specifies the nutational motion in the neighbourhood of this precession. The variables W, Θ and c are independent integrals of system (1.1) at $\varepsilon = 0$ and are connected with R, L, and E by the relations

$$R = \lambda W, \ L = \Omega_{20}\sin\Theta + \lambda W\cos\Theta \tag{2.8}$$
$$E = (\lambda W^2 + \Omega_{20}^2 + \omega_*^2 c^2)/2 + \varkappa\cos\Theta + O(c^3)$$

The change of variables reduces system (1.1) to a form that is convenient for the application of the averaging method /8/.

3. ***Averaged equations for the evolutionary variables.*** We shall analyse the perturbed motion using the method of averaging in the form developed by Volosov /9/. We shall assume that the variables W, Θ, and c are slow, and that the variables ν, ψ, and φ are fast.

The derivation of the equations for the evolutionary variables consists of making two changes of variables in succession

$$(\Omega_1, \Omega_2, \Omega_3, \vartheta, \psi, \varphi) \xrightarrow{1} (R, L, E, \nu, \psi, \varphi) \xrightarrow{2} (W, \Theta, c, \nu, \psi, \varphi).$$

in system (1.1). For R, L, and E we obtain the equations

$$R^{\cdot} = \varepsilon M_3, \ L^{\cdot} = \varepsilon(M_2\sin\vartheta + M_3\cos\vartheta) \tag{3.1}$$
$$E^{\cdot} = \varepsilon(\Omega_1 M_1 + \Omega_2 M_2 + \Omega_3 M_3)$$

We then express R, L, and E in (3.1) in terms of W, Θ and c using relations (2.8), and obtain

$$\lambda W^{\cdot} = \varepsilon\Phi_1, \quad \frac{\partial L}{\partial W} W^{\cdot} + \frac{\partial L}{\partial \Theta}\Theta^{\cdot} = \varepsilon\Phi_2 \tag{3.2}$$
$$\frac{\partial E}{\partial W} W^{\cdot} + \frac{\partial E}{\partial \Theta}\Theta^{\cdot} + \frac{\partial E}{\partial c} c^{\cdot} = \varepsilon\Phi_3; \quad \Phi_i = \Phi_i(W, \Theta, c, \nu, \psi, \varphi) \quad (i = 1, 2, 3)$$

Relations (3.2) are a system of linear equations with respect to the derivatives of the slow variables, whose determinant is

$$D = \lambda \frac{\partial L}{\partial \Theta}\frac{\partial E}{\partial c} = \pm\frac{c}{\sqrt{d}}[\lambda\omega_*^4\sin\Theta + O(c)]$$

The choice of sign in the latter relation is consistent with the choice of sign in (2.5). For $c \neq 0$ and $d \neq 0$, system (3.2) reduces to the form

$$W^{\cdot} = \varepsilon Z_1, \ \Theta^{\cdot} = \varepsilon Z_2, \ c^{\cdot} = \varepsilon Z_3 \tag{3.3}$$
$$Z_i = Z_i(W, \Theta, c, \nu, \psi, \varphi) \ (i = 1, 2, 3)$$

We shall average the right hand side of Eqs.(3.3) over Lagrangian motion with fixed slow variables. The averaging procedure is complicated by the unevenness of the variation of the fast variables ψ and φ because of the presence of the periodic component. However, arguments similar to those contained in /10/ enable us to establish that in the case of rational incommensurability of the frequencies $\omega_\vartheta, \omega_\psi$, and ω_φ averaging over time is equivalent to independent averaging over ν, ψ, and φ. Thus, for an arbitrary function

$f(W, \Theta, c, \nu, \psi, \varphi)$,

$$\langle f\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(W, \Theta, c, \omega_\nu t + \nu_0, \omega_\psi t + \psi_1(t), \omega_\varphi t + \varphi_1(t))\, dt$$
$$= \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f(W, \Theta, c, \nu, \psi, \varphi)\, d\nu\, d\psi\, d\varphi$$

from which the result of the averaging does not depend on the initial data ν_0, ψ_0, and φ_0. As a result we obtain an averaged first-order approximation system

$$W^{\cdot} = \varepsilon V_1,\ \Theta^{\cdot} = \varepsilon V_2,\ c^{\cdot} = \varepsilon V_3 \tag{3.4}$$
$$V_j = V_{j0}(W, \Theta) + c^2 V_{j2}(W, \Theta) + O(c^3)\ (j = 1, 2)$$
$$V_3 = c\,[V_{31}(W, \Theta) + O(c)]$$

in which previous notation now stands for the averaged variables.

Because the second change of variables does not affect the fast variables, in constructing system (3.4) the averaging can already be carried out in system (3.1), and then the change of variables $(R, L, E) \to (W, \Theta, c)$ can be made in the averaged system.

Additional analysis shows that Eqs.(3.4) also hold in the case $c = 0$. In the phase space of system (3.4) the condition $c = 0$ defines an integral manifold. Solutions lying on this manifold correspond to quasistatic evolution of regular precession without the excitation of nutational oscillations /11/. Because there are no terms linear in c in the first two equations of system (3.4), small nutational oscillations only weakly influence the behaviour of the variables W and Θ.

The conditions $V_{31}(W, \Theta) \lessgtr 0$ determine attractive and repulsive domains in a neighbourhood of the manifold $c = 0$. In the attractive domain the amplitude of small nutational oscillations decreases, and in the repulsive domain it increases. The characteristic time for changes in the amplitude of the oscillations is comparable with the characteristic time for changes in the variables W and Θ. In the course of the perturbed motion's evolution multiple shifts between growth and decay regimes for the nutational oscillations are possible if the solution passes from the repulsive domain to the attractive domain, and vice-versa.

We shall use the above method to analyse the perturbed solutions of system (1.1) in specific problems.

4. The influence of a small constant moment. We shall consider the evolution of a motion produced by the influence of a small moment, constant in a coupled coordinate system, with components $\varepsilon m_1, \varepsilon m_2, \varepsilon m_3$ (where $(\varepsilon \ll 1$ and $m_i = \text{const})$. In the case being considered system (3.1) can be written as follows:

$$R^{\cdot} = \varepsilon m_3,\ L^{\cdot} = \varepsilon\,[\sin\vartheta\,(m_1 \sin\varphi + m_2\cos\varphi) + m_3\cos\vartheta] \tag{4.1}$$
$$E^{\cdot} = \varepsilon\,[m_1(\Omega_1\cos\varphi + \Omega_2\sin\varphi) - m_2(\Omega_1\sin\varphi - \Omega_2\cos\varphi) + m_3\Omega_3]$$

Following the method given above, we construct an averaged system of equations of motion. Because the right hand sides of the equations of system (4.1) do not depend on the angle of precession ψ, averaging along the unperturbed motion reduces to independent averaging with respect to ν and φ. Using the relations

$$\langle\sin\vartheta\sin\varphi\rangle = \langle\sin\vartheta\cos\varphi\rangle = 0,\ \langle\Omega_j\cos\varphi\rangle = \langle\Omega_j\sin\varphi\rangle = 0$$
$$(j = 1, 2)$$
$$\langle\cos\vartheta\rangle = \cos\Theta + c^2 h + O(c^3),\ h = (3\varkappa\sin^2\Theta)/(2\omega_*^2) - \cos\Theta$$

we obtain

$$\lambda W^{\cdot} = \varepsilon m_3,\ \ \Theta^{\cdot} = \varepsilon m_3\left(\frac{\partial L}{\partial\Theta}\right)^{-1}\left(\cos\Theta - \frac{1}{\lambda}\frac{\partial L}{\partial W} + c^2 h + O(c^3)\right) \tag{4.2}$$
$$c^{\cdot} = -\frac{\varepsilon m_3}{\omega_*^2}\left\{\frac{\omega_*}{\lambda}\frac{\partial\omega_*}{\partial W} + \left(\frac{\partial L}{\partial\Theta}\right)^{-1}\left[\frac{\partial E_0}{\partial\Theta}h + \omega_*\frac{\partial\omega_*}{\partial\Theta}\left(\cos\Theta - \frac{1}{\lambda}\frac{\partial L}{\partial W}\right)\right] + O(c)\right\}c$$

Here $E_0 = E(W, \Theta, 0)$ is the total energy of the body for motion in the regime of regular precession $(c = 0)$. Because of their complexity we shall not give explicit expressions for the partial derivatives $\partial L/\partial W, \partial L/\partial\Theta, \partial\omega_*/\partial W, \partial\omega_*/\partial\Theta$, and $\partial E_0/\partial\Theta$ found in (4.2).

The system of equations obtained does not contain m_1 and m_2, and so the evolution of the motion is solely determined by the projections of the perturbing moment onto the body's axis of dynamical symmetry.

For $c=0$ the trajectories of system (4.2) satisfy the integral relation $2\varkappa\cos\Theta+\Omega_{20}^2=$ const and fill the domain $U=\{(W,\Theta):0<\Theta<\pi,\ d(W,\Theta)>0\}$ in the (W,Θ) plane. (Because of (2.5) and (3.3) the condition $d(W,\Theta)>0$ is a condition for the physical achievability of the precession and the solubility of Eq.(3.3)). Depending on the choice of sign in (2.5), two types of phase portrait are possible for system (4.2) on the integral manifold $c=0$. With the help of identifications of segments of the boundaries $d(W,\Theta)=0$ of phase portraits of different types and the continuation of trajectories as solutions of differential equations, we obtain a complete phase portrait showing the general picture of motion on the manifold.

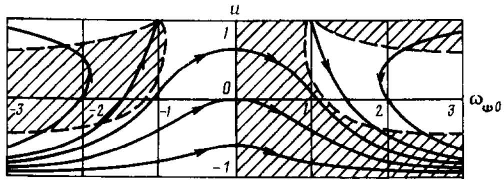

Fig.1

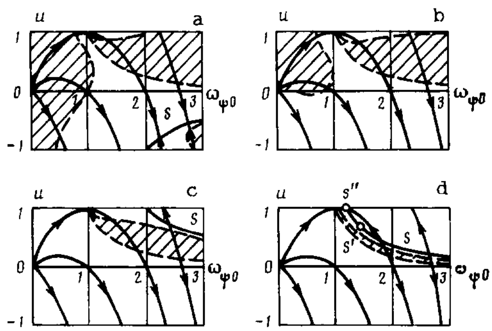

Fig.2

It is convenient to present the complete phase portrait in a more traditional form, mapping it onto the plane $(\omega_{\psi0}, u)$ where $\omega_{\psi0}=\Omega_{20}/\sin\Theta$ is the angular velocity of the regular precession and $u=\cos\Theta$. To make the mapping single-valued the choice of sign in expression (2.5) for $\Omega_{20}(W,\Theta)$ should be consistent with the form of the corresponding phase portrait. Fig.1 shows the complete phase portrait of system (4.2) in the $(\omega_{\psi0}, u)$ plane for the case $\lambda={}^5/_4$, $\varkappa=1$ and $m_1<0$. The diagonal bars on the phase portrait show the repulsive domain of the integral manifold $c=0$.

In the case being considered the first approximation of the averaging method can have resonances when $\omega_\varphi=l\omega_\vartheta$ $(l=0,\pm1,\pm2,\ldots)$. Investigation of the resonances lies outside the remit of this paper. The results of direct numerical integration of system (1.1) confirm the presence of characteristically resonant effects for $\omega_\varphi\approx0$ and for $|\omega_\varphi|\approx\omega_\vartheta$. For example, according to /12/, trajectories with initial conditions that only differ by the value of the angle of proper rotation, diverge by an amount of order $\sqrt{\varepsilon}$ after going through a resonance.

5. ***The motion of a Lagrange gyroscope with a cavity filled with highly viscous fluid.*** As was established in /13/, the influence of a highly viscous fluid on the motion of a rigid body is equivalent to the action on the frozen system of an external moment

$$m_1=\varepsilon\sum_{i=1}^{3}[P_{1i}b_i+(\omega_2P_{3i}-\omega_3P_{2i})a_i]\quad(123)\tag{5.1}$$

where $\|P_{ij}\|_{i,j=1}^3$ is a constant symmetric positive matrix depending on the shape and dimensions of the cavity; a_i and b_i $(i=1,2,3)$ are respectively equal to the derivatives $\omega_i^{\cdot}$ and $\omega_i^{\cdot\cdot}$, computed using the unperturbed equations of motion $(\varepsilon=0)$. The small parameter ε is a dimensionless combination of parameters from the body, cavity and fluid.

Following the above method, we obtain for the slow variables an averaged system of first order of approximation:

$$\lambda W^{\cdot} = \varepsilon \left(Q_1 + c^2 Q_2 + O(c^3)\right) \tag{5.2}$$

$$\Theta^{\cdot} = -\frac{\varepsilon}{\lambda}\left(\frac{\partial L}{\partial \Theta}\right)^{-1}\left(\frac{\partial L}{\partial W}\right)\left(Q_1 + c^2 Q_2 + O(c^3)\right)$$

$$c^{\cdot} = -\frac{\varepsilon}{\lambda \omega_*^2}\left(\frac{\partial L}{\partial \Theta}\right)^{-1}\left[\frac{D(E_0, L)}{D(W, \Theta)} Q_1 + \omega_* \frac{D(\omega_*, L)}{D(W, \Theta)} Q_2 - \frac{\partial L}{\partial \Theta} Q_3 + O(c)\right] c$$

$$Q_1(W, \Theta) = (P_{11} - P_{22})\Omega_{20}\left[(\lambda - 1) W \Omega_{20} - \varkappa \sin \Theta\right]/2$$

$$Q_2(W, \Theta) = (P_{11} + P_{22})\Omega_{20}\left[(\lambda - 1)\omega_*^2 + (2 - \lambda)\varkappa h\right]/2$$

$$Q_3(W, \Theta) = (P_{11} + P_{22})(2hE_0 + \omega_*^2 \cos \Theta - \varkappa)/2 - (\lambda - 1) W Q_2$$

System (5.2) has the first integral $L(W, \Theta) = \mathrm{const}$, expressing the constancy of the projection of the total kinetic moment of the body and fluid onto the vertical.

The analysis shows that the phase portrait of the system in question on the integral manifold $c = 0$ after the change of variables $(W, \Theta) \to (\omega_{\psi 0}, u)$ is symmetrical about the axis $\omega_{\psi 0} = 0$. The behaviour of the system's trajectories on the manifold depends significantly on which of the interval $(0, 1)$, $(1, {}^4/_3)$, $({}^4/_3, 2)$ contains the ratio of the moments of inertia λ. Examples of phase portraits of various kinds are given in Fig.2, a-d, where $\lambda = {}^3/_4, 1, {}^5/_4$, and ${}^7/_8$ respectively. Because of the symmetry we need only consider the domain of positive values for $\omega_{\psi 0}$ in the phase portrait.

For $\lambda \neq 1$ the system of Eqs.(5.2) has stationary solutions in which $c = 0$ while the variables W and Θ are connected by the relation $(\lambda - 1)\omega_{\psi 0}^2 (W, \Theta) = \varkappa / \cos \Theta$, defining a one-parameter family of permanent rotations of the body and fluid as a single body (curve S on the phase portraits in Fig.2).

It should be noted that in the neighbourhood of the permanent rotation $\omega_{\psi} \approx 0$ and the averaged system (5.2) is formally inapplicable (because of the exclusion of the case of a cavity with axis of symmetry parallel to the axis of symmetry of the body). However, the existence of these stationary motion regimes of a body with a viscous fluid can be proved directly for system (1.1).

All permanent rotations are stable for the variables W, Θ and c when $0 < \lambda < 1$ and unstable when $1 < \lambda \leqslant {}^4/_3$. If ${}^4/_3 < \lambda < 2$ then only permanent rotations lying on the segment $s's''$ of the curve S in Fig.2d are stable.

In the neighbourhood of permantent rotation, the variable c in the linear approximation evolves independently of the variables W and Θ. The characteristic time for the evolution is $T_1 = |\rho_1|^{-1}$ for the variable c and $T_2 = |\rho_2|^{-1}$ for the variables W and Θ, where

$$\rho_1 = \varepsilon V_{31}(W, \Theta)\Big|_S = -\frac{\varepsilon}{\lambda \omega_*^2}\left(\frac{\partial L}{\partial \Theta}\right)^{-1}\left(\frac{D(\omega_*, L)}{D(W, \Theta)}\omega_* Q_2 - \frac{\partial L}{\partial \Theta} Q_3\right)\Big|_S \tag{5.3}$$

$$\rho_2 = \frac{\varepsilon}{\lambda}\left(\frac{\partial L}{\partial \Theta}\right)^{-1}\frac{D(Q_1, L)}{D(W, \Theta)}\Big|_S = -\frac{\varepsilon \omega_{\psi 0}^2 \varkappa}{\lambda u \omega_*^2}(P_{22} + P_{33})(1 - u^2)\left[3(\lambda - 1)u^2 - 1\right]\Big|_S$$

are the roots of the characteristic equation of the linearized system (5.2). Another root of this system is $\rho_3 = 0$.

As well as permanent rotations, there are trivial stationary regimes in which the body and fluid rotate uniformly about the axis of dynamical symmetry directed along the vertical. The trivial regimes can be represented by points at the upper and lower boundaries of the phase portrait. The behaviour of trajectories of system (5.2) near the boundaries can be found using the results of investigations of the stability of the trivial regimes by Lyapunov methods /14/.

Properties of the perturbed motion established by considering the averaged system (5.2) are in good agreement with the results of numerical integration of the original system of Eqs.(1.1).

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THE LOCAL BOUNDEDNESS OF THE PERTURBED MOTIONS OF A GYROSCOPE IN GIMBALS WITH DISSIPATIVE AND ACCELERATING FORCES*

S.A. BELIKOV

The motion of an unbalanced gyroscope in gimbals in a central Newtonian field of forces is considered, taking the masses of the suspension rings into account. It is assumed that there is a moment of forces of viscous friction acting on the axis of rotation of one of the rings, and there is an accelerating (electromagnetic) moment applied to the axis of rotation axis of the other ring. The equations of motion have a partial solution such that the mean velocity of the outer ring is perpendicular to the direction from the centre of gravitation S to the stationary point O, the middle plane of the inner ring contains this direction, and the gyroscope rotates about SO with an arbitrary constant angular velocity.

The equations of perturbed motions of the system in the neighbourhood of the corresponding state of equilibrium are obtained to within terms of order three. The characteristic equation of the system is considered and the coefficients of the equation are found in the region F_0 of admissible values of the parameters. The question of the distribution of eigenvalues with respect to the imaginary axis is studied. A region in F_0 is constructed in which the pairs of complex conjugate eigenvalues have small real parts among which there are some positive ones, and the absolute values of the resonance mistuning between the imaginary parts are not small. In this region we obtain sufficient conditions for local uniform boundedness of perturbed motions of the gyroscope in gimbals with dissipative and accelerating forces with respect to the partial solution mentioned above. These conditions are found in the form of constraints for the coefficients of the normal form and, eventually, for the original parameters of the system and the real and imaginary parts of the eigenvalues. To provide illustrative interpretation, some special cases are considered and the regions of

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